

# Lower Bounds in Algebraic Complexity via Symmetry and Homomorphism Polynomials

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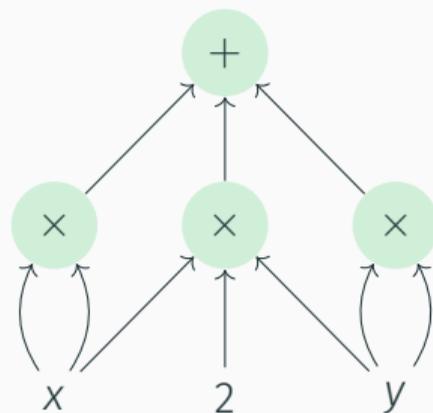
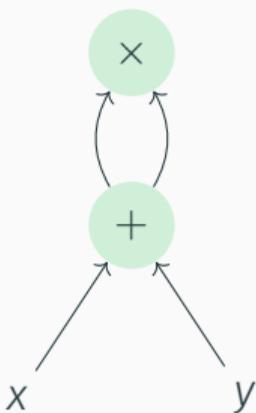
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Joint work with Prateek Dwivedi and Tim Seppelt (IT-Universitetet i København)

# Algebraic complexity

The *complexity* of a polynomial is the size of the smallest algebraic circuit representing it.



## Determinant

$$\det_n = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^n x_{i,\pi(i)}$$

has poly-size algebraic circuits.

## Permanent

$$\text{perm}_n = \sum_{\pi \in S_n} \prod_{i=1}^n x_{i,\pi(i)}$$

### VP vs. VNP (Valiant (1979))

Does  $\text{perm}_n$  admit poly-size algebraic circuits?

## Towards Valiant's conjecture

**Approach:** Prove lower bounds for **restricted** circuit models.

No sub-exponential size family of *monotone* circuits computes the permanent.

[Jerrum, Snir 1982]

No sub-exponential size family of *depth 3* circuits computes the permanent.

[Grigoriev, Karpinski 1998]

**But:** Both methods yield similar lower bounds for the determinant.

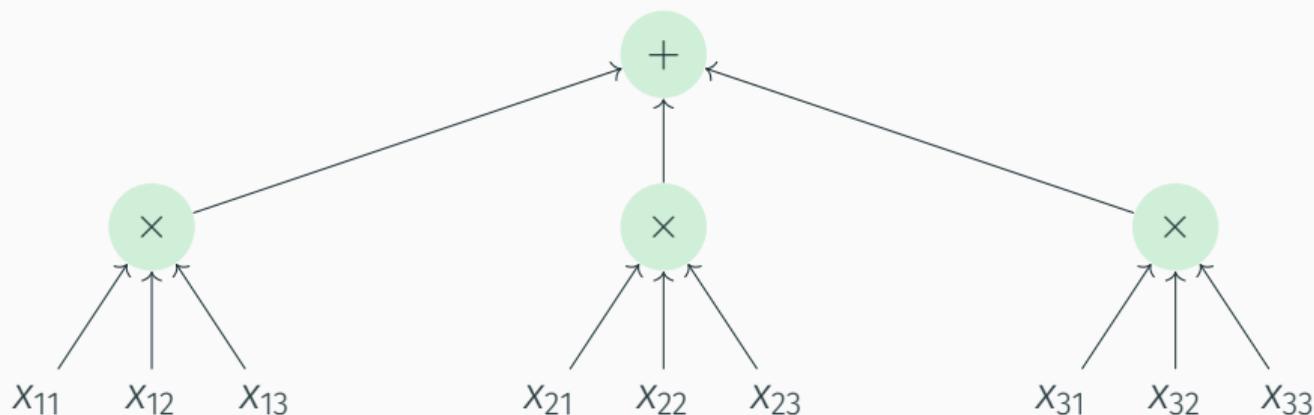
**Theorem (Dawar, Wilsenach 2020)**

*There are no subexponential-size symmetric circuits for  $\text{perm}_n$ .*

*There are polynomial-size symmetric circuits for  $\text{det}_n$ .*

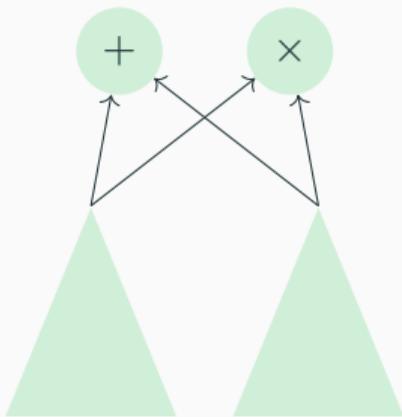
# Symmetric circuits

- We consider polynomials in variables  $x_{ij}$  for  $i, j \in [n]$ .
- A polynomial / circuit is *symmetric* if it is invariant under the action of  $\mathbf{Sym}_n \times \mathbf{Sym}_n$ .



symVP

symmetric **circuits** of  
polynomial orbit-size

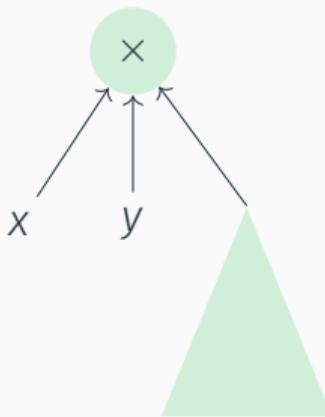


VP

poly-size **circuits**

symVBP

symmetric **skew circuits**  
of polynomial orbit-size

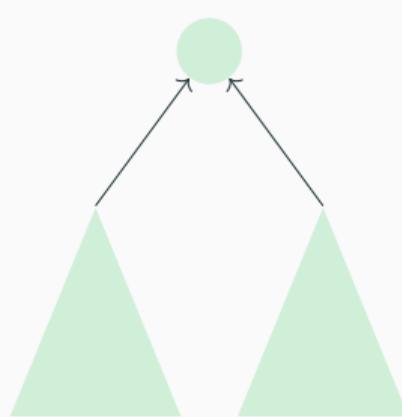


VBP

poly-size **skew circuits**

symVF

symmetric **formulas** of  
polynomial orbit-size



VF

poly-size **formulas**

Open problem:

Do the algebraic complexity classes VF, VBP, VP form a **strict** hierarchy?

**Theorem**

$$\text{symVF} \subsetneq \text{symVBP} \subsetneq \text{symVP}.$$

# A Symmetric Algebraic Complexity Theory

VNP

UI

VP

UI

VBP

UI

VF

symVP

bounded treewidth

U†

U†

symVBP

bounded pathwidth

U†

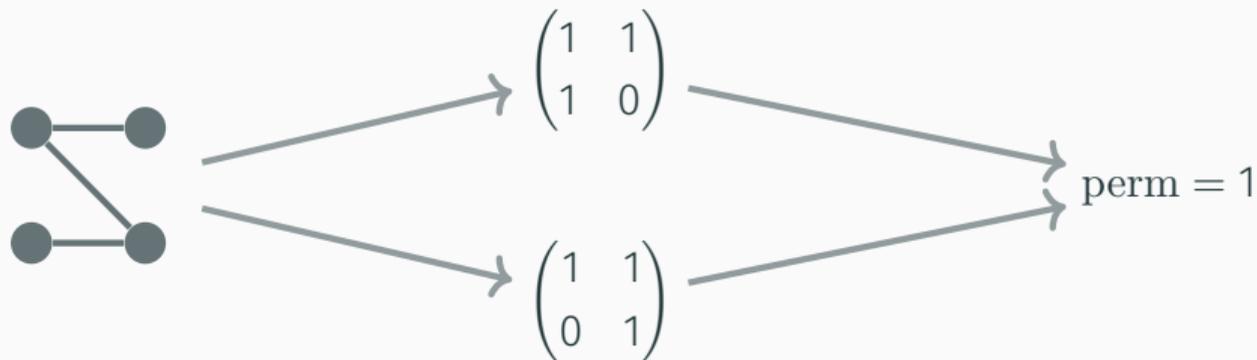
U†

symVF

bounded treedepth

# Graph-theoretic semantics of symmetric polynomials

Symmetric polynomials are functions of  $(n, n)$ -vertex bipartite graphs. E.g., the permanent  $\text{perm}_n(G)$  is the number of perfect matchings in  $G$ .



For a bipartite multigraph  $F$  and  $n \in \mathbb{N}$ ,

$$\text{hom}_{F,n} := \sum_{h: V(F) \rightarrow [n]} \prod_{uv \in E(F)} x_{h(u), h(v)}$$

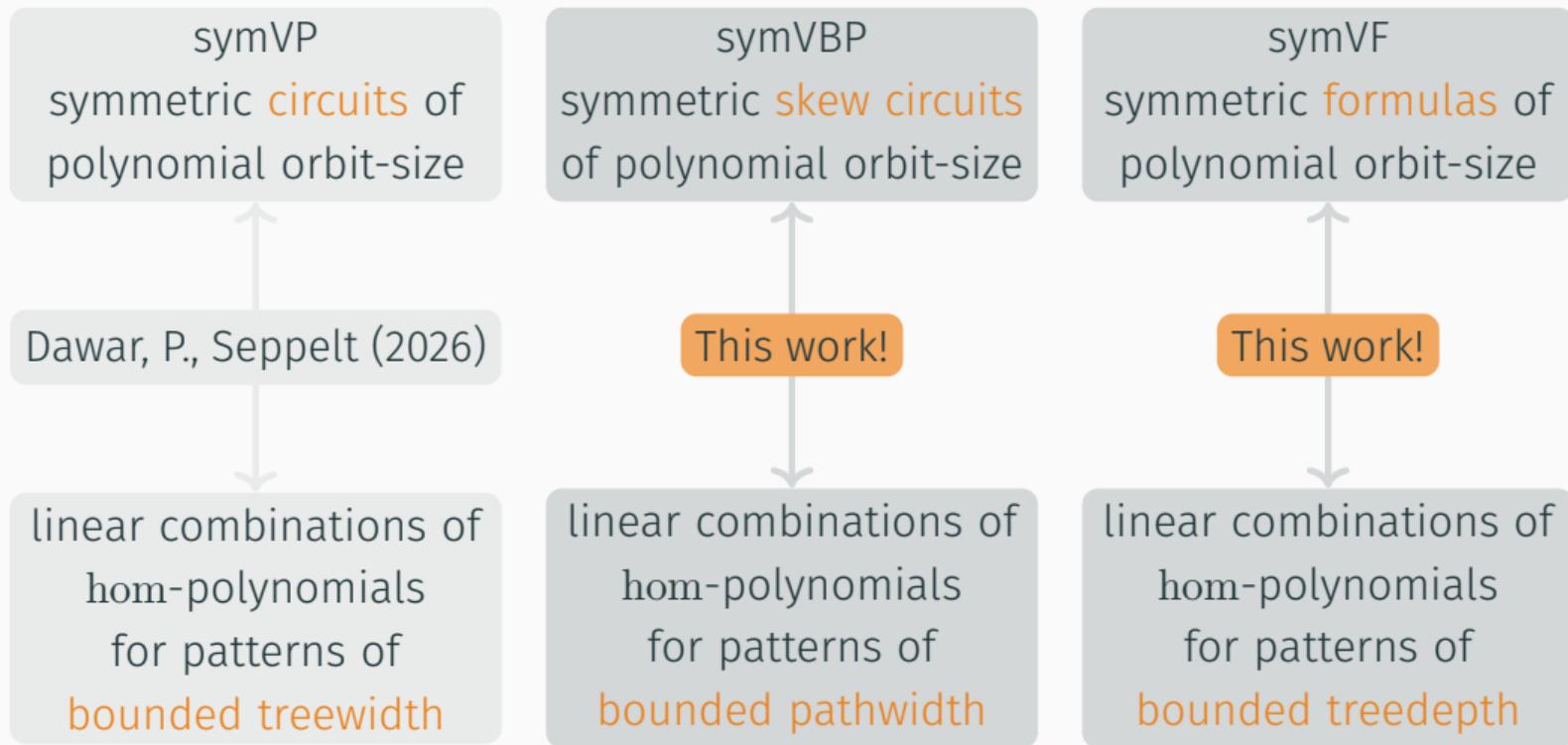
### Fact

A polynomial is *symmetric*  $\iff$  it is a linear combination of *hom-polynomials*.

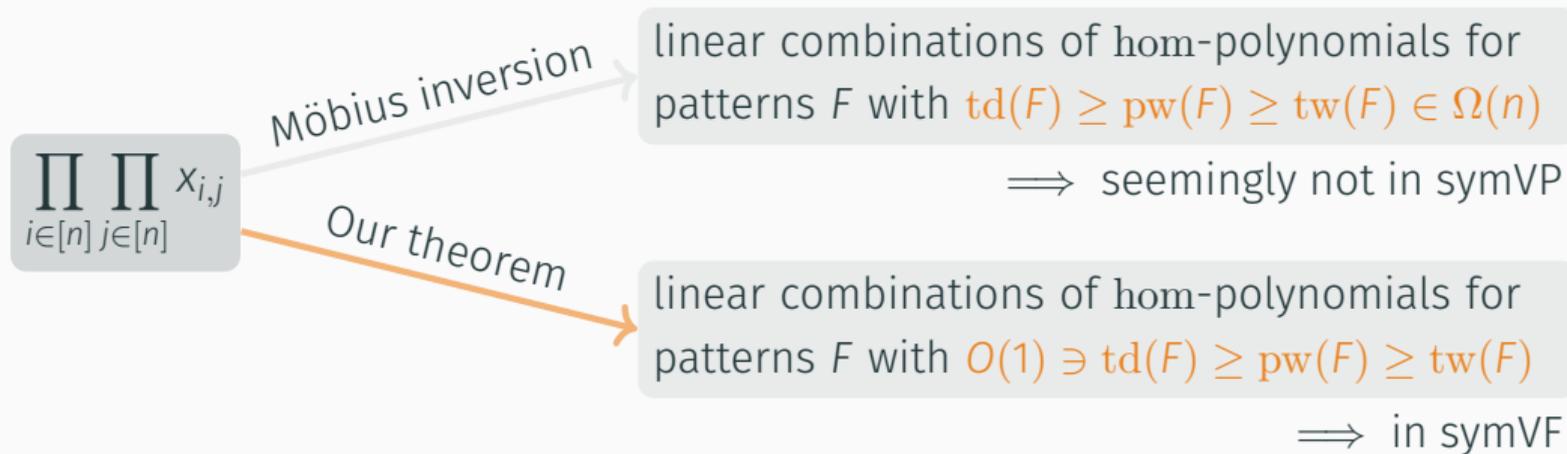
### Corollary

Every *graph parameter* can be written as

$$\rho_n(\star) = \sum \alpha_{F,n} \text{hom}(F, \star).$$



The same polynomial may have multiple representations as linear combinations of hom-polynomials!



Separating symVF, symVBP, and symVP relies on substantial work on **homomorphism indistinguishability**.

Cai, Fürer, & Immerman (1992); Dawar & Wang (2017); Roberson (2022); Neuen (2024); Dell, Grohe, & Rattan (2018); Dvořák (2010); Montacute & Shah (2024); Grohe (2020); Fluck, Spitzer, Seppelt (2024); Seppelt (2024); Dawar, P., Seppelt. (2026)

## Theorem

1. *symVP contains **VP-complete polynomials**, namely  $(\text{hom}_{F_n, n})$  for  $\text{tw}(F_n) \in O(1)$ ,  $\text{pw}(F_n) \geq \Omega(\log n)$  (for example binary trees).*
2. *symVBP contains **VBP-complete polynomials**, namely  $(\text{hom}_{F_n, n})$  for  $\text{pw}(F_n) \in O(1)$ ,  $\text{td}(F_n) \geq \Omega(\log n)$  (for example paths).*
3. *If  $(p_n)$  is a linear combination of  $\text{hom}$ -polynomials of pattern graphs of size at most  $n^{1-\varepsilon}$ , then  $(p_n) \in \text{VP} \iff (p_n) \in \text{symVP}$  (assuming  $\text{VFPT} \neq \text{VW}[1]$ ).*

Every graph parameter can be written as **linear combinations of homomorphism counts** and these linear combinations convey complexity-theoretic information.

- Unconditional symmetric circuit lower bounds.
- For polynomials of **sublinear pattern size**, **symmetric** and **non-symmetric** algebraic computation have the same power.
- Next: What about patterns of linear size?