Lower Bounds in Algebraic Complexity via Symmetry and Homomorphism Polynomials

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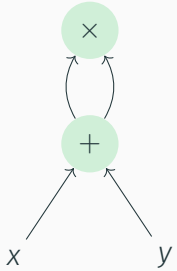
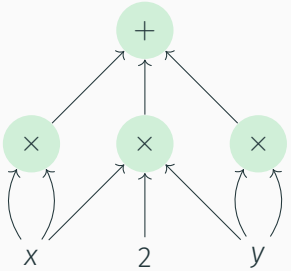
Algebraic complexity

The *complexity* of a polynomial is the size of the smallest algebraic circuit representing it.





VP vs. VNP

Determinant

$$\det_n = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} X_{i,\pi(i)}$$

has poly-size algebraic circuits.

Permanent

$$\operatorname{perm}_n = \sum_{\pi \in S_n} \prod_{i=1}^{n} X_{i,\pi(i)}$$

VP vs. VNP (Valiant, 1979)

Does perm_n admit poly-size algebraic circuits?

Towards Valiant's conjecture

Approach: Prove lower bounds for **restricted** circuit models.

No sub-exponential size family of *monotone* circuits computes the permanent.

[Jerrum, Snir 1982]

No sub-exponential size family of *depth 3* circuits computes the permanent.

[Grigoriev, Karpinski 1998]

But: Both methods yield similar lower bounds for the determinant.

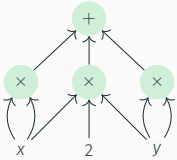
Theorem (Dawar, Wilsenach 2020)

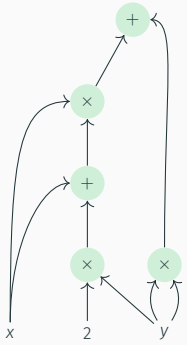
There are no subexponential-size ***symmetric circuits*** *for* perm_n.
There are polynomial-size ***symmetric circuits*** *for* $\det_n$.

Symmetry in polynomials and circuits

The polynomial $(x + y)^2$ is symmetric under exchanging x and y.

A *symmetric circuit* is one with an *automorphism* swapping x and y.





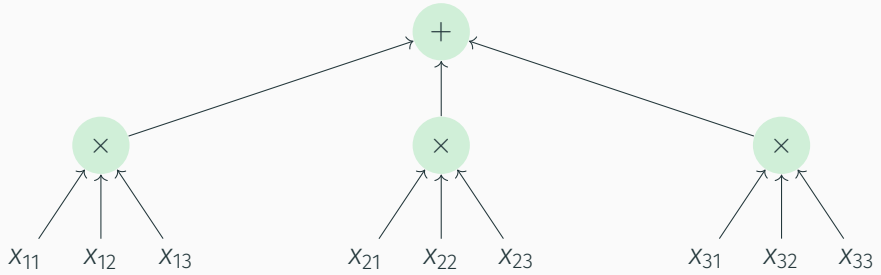
Matrix-symmetric polynomials and circuits

For a *matrix polynomial* p over $X_n = \{x_{ij} \mid 1 \leq i, j \leq n\}$, we consider $\mathbf{Sym}_n \times \mathbf{Sym}_n$-symmetry.

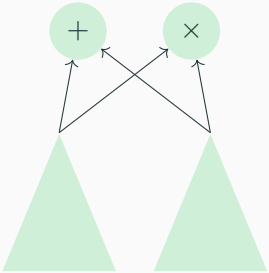
Each $(\pi, \sigma) \in \mathbf{Sym}_n \times \mathbf{Sym}_n$ maps x_{ij} to $x_{\pi(i)\sigma(j)}$.

Matrix-symmetric polynomials and circuits

A polynomial / circuit over X_n is *matrix-symmetric* if it is invariant under the action of $\mathbf{Sym}_n \times \mathbf{Sym}_n$.

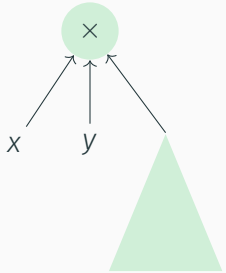


symVP
symmetric circuits of polynomial orbit-size



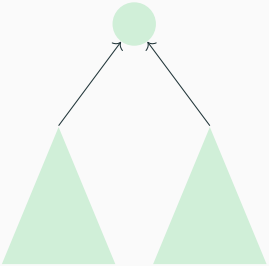
VP
poly-size circuits

symVS
symmetric skew circuits of polynomial orbit-size



VS
poly-size skew circuits

symVF
symmetric formulas of polynomial orbit-size



VF
poly-size formulas

Main result

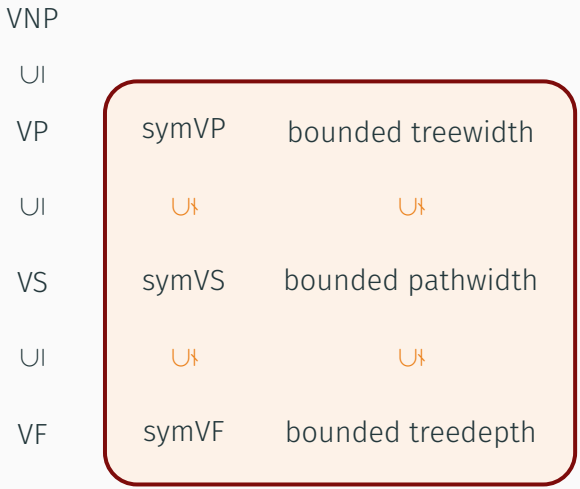
Open problem:

Do the algebraic complexity classes VF, VS, VP form a strict hierarchy?

Theorem

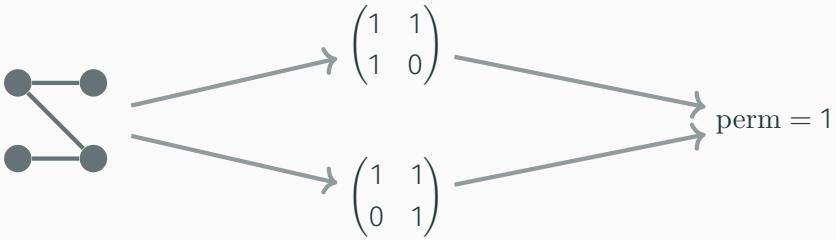
$$\mathrm{symVF} \subsetneq \mathrm{symVS} \subsetneq \mathrm{symVP}.$$

A Symmetric Algebraic Complexity Theory



Graph-theoretic semantics of symmetric polynomials

Symmetric polynomials are functions of (n, n)-vertex bipartite graphs.
E.g., the permanent $\mathrm{perm}_n(G)$ is the number of perfect matchings in G.



For a bipartite multigraph F and $n \in \mathbb{N}$,

$$\hom_{F,n} \coloneqq \sum_{h\colon V(F)\to[n]\uplus[n]} \prod_{uv\in E(F)} X_{h(u),h(v)}$$

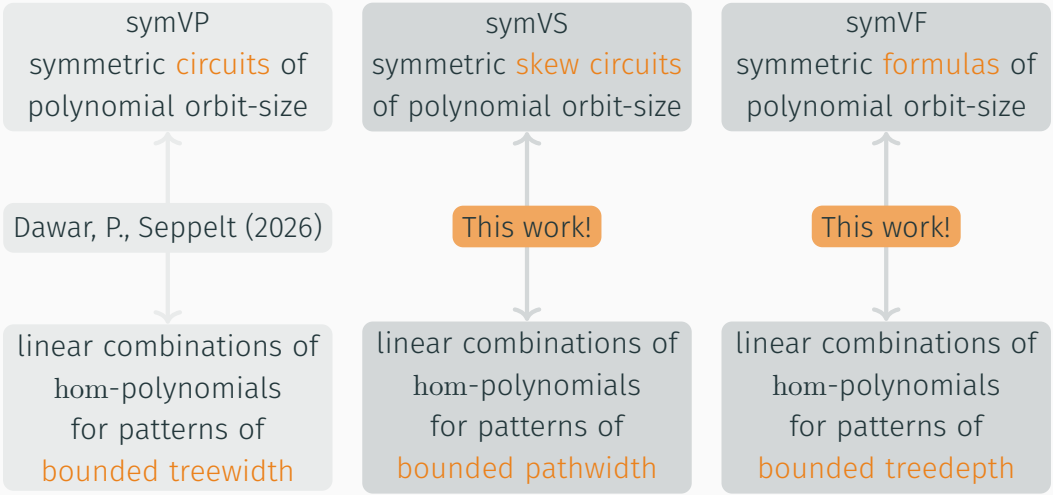
Fact

A polynomial is symmetric $\iff$ it is a linear combination of $\hom$-polynomials.

Corollary

Every graph parameter can be written as

$$p_n(\star) = \sum \alpha_{F,n} \hom(F,\star).$$


symVP
symmetric circuits of
polynomial orbit-size
Dawar, P., Seppelt (2026)
linear combinations of
hom-polynomials
for patterns of
bounded treewidth
symVS
symmetric skew circuits
of polynomial orbit-size
This work!
linear combinations of
hom-polynomials
for patterns of
bounded pathwidth
symVF
symmetric formulas of
polynomial orbit-size
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Proof of the separation result

Theorem

$$\text{symVF} \subsetneq \text{symVS} \subsetneq \text{symVP}.$$

Proof.

E.g. for symVS $\subsetneq$ symVP, show that there exist graphs (F_n) of bounded treewidth such that $\hom_{F,n}$ is not a linear combination of $\hom$-polynomials of bounded pathwidth.

$\Rightarrow$ via theory of ***homomorphism indistinguishability***. $\square$

Strength of the lower bounds

The power of the symmetric circuit models

Theorem

1. symVP *contains* ***VP-complete polynomials****, namely* $(\mathrm{hom}_{F_n,n})$ *for* $\mathrm{tw}(F_n) \in O(1), \mathrm{pw}(F_n) \geq \Omega(\log n)$ *(for example binary trees).*
2. symVS *contains* ***VS-complete polynomials****, namely* $(\mathrm{hom}_{F_n,n})$ *for* $\mathrm{pw}(F_n) \in O(1), \mathrm{td}(F_n) \geq \Omega(\log n)$ *(for example paths).*

Does symmetric VP capture all of VP?

Theorem

If (p_n) is a linear combination of hom*-polynomials of pattern graphs of size at most $n^{1-\varepsilon}$, then $(p_n) \in \mathsf{VP} \iff (p_n) \in \mathsf{symVP}$ (assuming VFPT $\neq$ VW[1]).*

Proof.

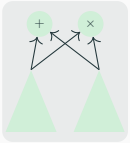
- Assuming VFPT $\neq$ VW[1], hom-polynomials of unbounded treewidth do not admit poly-size circuits.
- $\Rightarrow$ If (p_n) is a linear combination of hom-polynomials of unbounded treewidth, it is not in VP. Else, it is even contained in symVP.

□

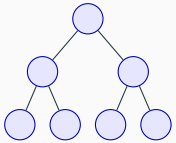
Outlook: Applications in proof complexity?

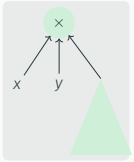
- In the **Ideal Proof System** (IPS), proofs are algebraic circuits.
- Can we use the homomorphism-framework to show lower bounds for symmetric IPS proofs?
- So far, we only have upper bounds [Dawar, Grädel, Kullmann, P. 2025].
- **Challenge:** Design hard symmetric instances in variables (x_{ij}).

Wrap-up

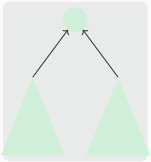


symVP ≈ treewidth





symVS ≈ pathwidth



symVF ≈ treedepth