# Arity Hierarchies for Quantifiers closed under Partial Polymorphisms

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# Generalized quantifiers

*Generalized quantifiers*, also called *Lindström quantifiers* (1966) are to first-order logic what **oracles** are to Turing machines.

**Definition**

Let $\tau = \{R_1, \dots, R_m\}$ and $\mathcal{K}$ be a class of $\tau$-structures. Then $\mathbf{Q}_{\mathcal{K}}$ is the *generalized quantifier* defining $\mathcal{K}$. Its *arity* is the maximum arity of any relation in $\tau$.

**Usage:** Let $\Psi = (\psi_{R_1}(\bar{y}), \dots, \psi_{R_m}(\bar{y}))$ be a tuple of $\sigma$-formulas, one for each $R_i \in \tau$.

Let $\mathbf{A}$ be a $\sigma$-structure. Then $\mathbf{A} \models Q_{\mathcal{K}}\bar{y}.\Psi$ if and only if $\underbrace{\Psi(\mathbf{A}) \in \mathcal{K}}_{=(A,(\psi_{R_i}^{\mathbf{A}})_{R_i \in \tau})}$.

**Example (Counting quantifiers)**

Express the **counting quantifier** $\exists^{=t}x$ as a (unary) generalized quantifier $Q_{\mathcal{K}}$:
Let $U$ be a unary relation symbol, and $\mathcal{K}$ be the class of all $\{U\}$-structures with $|U| = t$.

$$\exists^{=t}x.\varphi(x) \equiv Q_{\mathcal{K}}x.(\varphi(x)).$$

# Studying extensions of FO through the lens of generalized quantifiers

**Goal:** Systematically understand and compare expressive power of different extensions of FO.

$\Rightarrow$ Define *classes* $\mathcal{Q}$ of *generalized quantifiers*, and study $\mathcal{L}(\mathcal{Q})$, the extension of FO with all quantifiers in $\mathcal{Q}$.

**Classic example:** Let $\mathcal{Q}_r$ be the class of all quantifiers of *arity* $\leq r$.

**Theorem (Hella, 1992)**

*For every $r \in \mathbb{N}$,*

$$\mathcal{L}(\mathcal{Q}_r) \lneq \mathcal{L}(\mathcal{Q}_{r+1}).$$

Another way of structuring generalized quantifiers is by *closure conditions*:

- For *every* generalized quantifier $Q_{\mathcal{K}}$, $\mathcal{K}$ is closed under isomorphisms.
- Closure under linear-algebraic relaxations of isomorphism yields *linear-algebraic logic* ($\approx$ *rank logic*).
- [Dawar, Hella, CSL'24]: Quantifiers closed under partial polymorphisms.

**Definition**

An $\ell$-ary *polymorphism* from a structure $\mathbf{A}$ to a structure $\mathbf{B}$ is a homomorphism from $\mathbf{A}^\ell$ to $\mathbf{B}$.

Polymorphisms are used in the complexity classification of constraint satisfaction problems (CSP).

Identities satisfied by polymorphism $p$ of the solution domain determine the complexity:

- **Near-unanimity:** $p(x, x, \dots, x, y) = p(x, x, \dots, y, x) = p(x, y, \dots, x, x) = p(y, x, \dots, x, x) = x$
  $\Rightarrow$ CSP has bounded width (very easy).
- **Maltsev:** $p(x, x, y) = p(y, x, x) = y$
  $\Rightarrow$ CSP in PTIME (admits basis computation).

We consider *partial* near-unanimity/Maltsev polymorphisms:
Only defined when the arguments match the pattern of the identities.

## Logics with generalized quantifiers closed under partial polymorphisms

| $\mathcal{L}(\mathcal{Q}_r^{\mathcal{N}_\ell})$ | $\mathcal{L}(\mathcal{Q}_r^{\mathcal{M}})$ |
|---|---|
| FO with all quantifiers of arity $\leq r$ closed under $\ell$-ary partial **near-unanimity** polymorphisms | FO with all quantifiers of arity $\leq r$ closed under partial **Maltsev** polymorphisms |

- CSPs with near-unanimity/Maltsev polymorphisms can be solved in these logics, but much more... (even NP-complete problems)
- [Dawar, Hella '24] introduced games for these logics and proved an **arity hierarchy** for $\mathcal{L}(\mathcal{Q}^{\mathcal{N}_\ell})$ in terms of $\ell$.
- **In this work:** Interplay between $\ell$ and $r$?

**Theorem (CSL'26)**

*For every $r \geq 3$,*

$$\mathcal{L}(\mathcal{Q}_r^{\mathcal{N}_r}) \equiv \mathcal{L}(\mathcal{Q}_{r-1}).$$

Do the quantifiers $\mathcal{Q}_r^{\mathcal{N}_\ell}$ ever give us something else than *all* generalized quantifiers of a fixed arity?

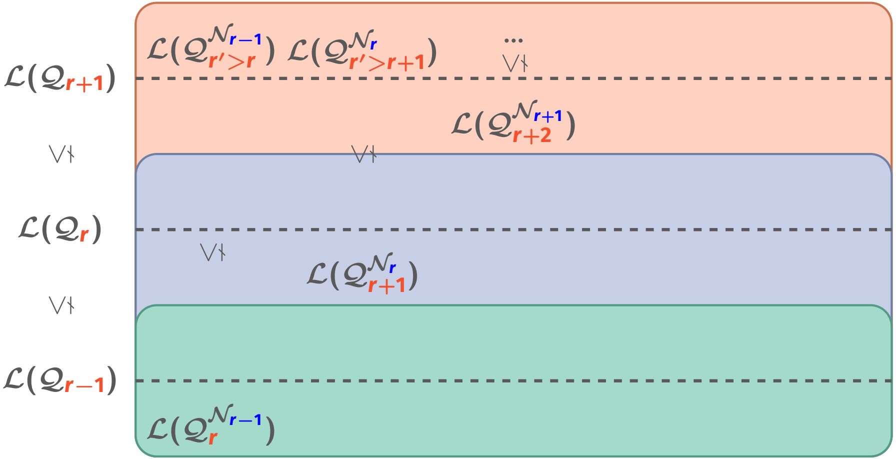
*They do!*

**Theorem (CSL'26)**

*For every $r \geq 3$,*

$$\mathcal{L}(\mathcal{Q}_{r-1}) \lneq \mathcal{L}(\mathcal{Q}_{r+1}^{\mathcal{N}_r}) \lneq \mathcal{L}(\mathcal{Q}_r).$$

# Arity hierarchies



# A note on Maltsev-closed quantifiers

- For partial **Maltsev**-closed quantifiers, an arity hierarchy follows from [Hella, '92]: $\mathcal{L}(\mathcal{Q}_r^{\mathcal{M}}) \lneq \mathcal{L}(\mathcal{Q}_{r+1}^{\mathcal{M}})$.
- **Open problem:** Separate $\mathcal{L}(\mathcal{Q}_r^{\mathcal{M}})$ from $\mathcal{L}(\mathcal{Q}_r)$.
- This requires a better idea than the usual *Cai-Fürer-Immerman* constructions because these encode problems over Abelian groups having a Maltsev polymorphism.
- **Partial success:**

**Theorem (CSL'26)**

*The k-variable fragments $\mathcal{L}^k$ are separated:*

$$\mathcal{L}^k(\mathcal{Q}_k^{\mathcal{M}}) \lneq \mathcal{L}^k(\mathcal{Q}_k).$$