Optimal Lower Bounds for Symmetric Modular Circuits

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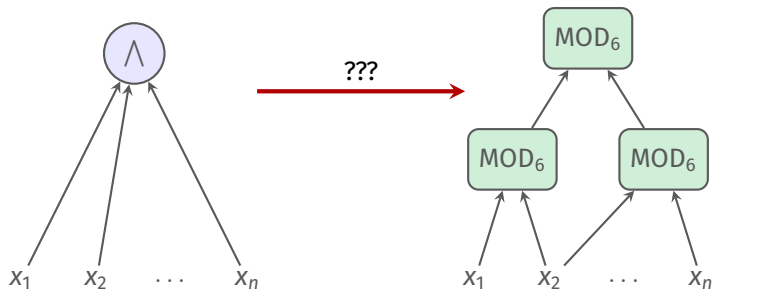
The CC^0 versus ACC^0 question

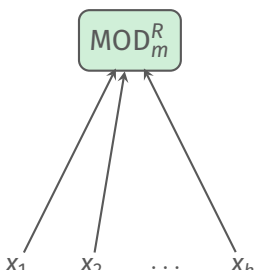
Can circuits with modular counting gates alone simulate Boolean operations efficiently?

(Barrington, Straubing, and Thérien 1990)



Boolean AND Circuit **Boolean MOD_6 Circuit**

Modular counting gates



Definition (Boolean modular gates)

Let $m \in \mathbb{N}, R \subseteq \mathbb{Z}_m$.

$$\mathrm{MOD}_m^R(x_1, \cdots, x_k) = \begin{cases} 1 & \text{if } (\sum x_i \mod m) \in R \\ 0 & \text{otherwise} \end{cases}$$

Why ask the question?

Definition

$\mathbf{CC^0}$: The class of all Boolean functions computable by families of polynomial-size depth-h MOD_m-circuits, for constant $h, m \in \mathbb{N}$.
$\mathbf{ACC^0}$: Like CC^0 but with $\{\mathrm{MOD}_m, \mathrm{AND}\}$-circuits.

- Determining the CC^0-complexity of AND_n would resolve the circuit complexity question "$\mathrm{CC}^0 = \mathrm{ACC}^0$?"

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- Upper bounds can be used to construct better locally-decodable codes. (Grolmusz 2000; Efremenko 2012)

Why should AND not be easy?

Conjecture ("Exponential Size Hypothesis")

For every depth h and modulus m, there exists $0 < \varepsilon \leq 1$ such that the $\mathrm{CC}_h[m]$-circuit complexity of AND_n is at least $2^{\Omega(n^\varepsilon)}$.

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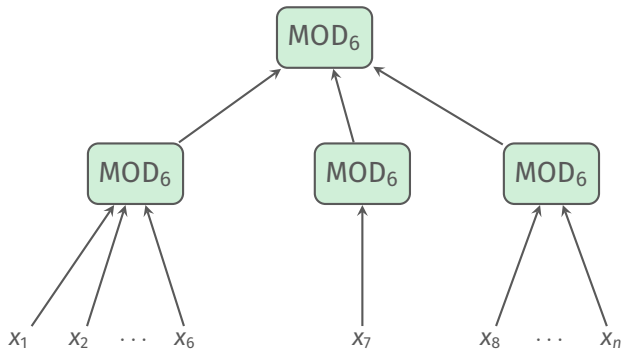

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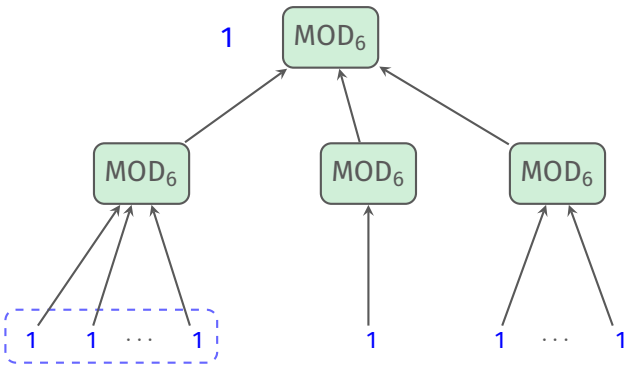

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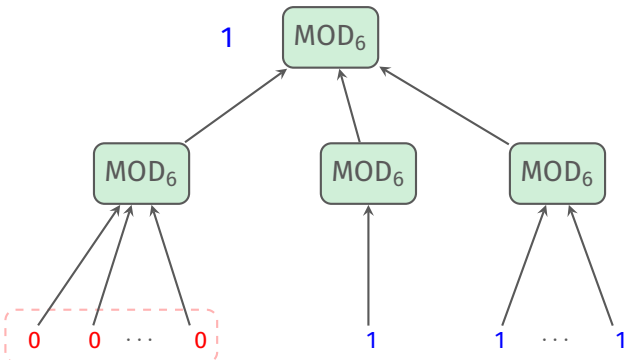


How not to express AND_n

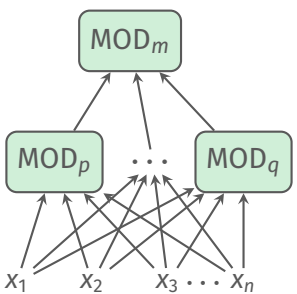
Upper bounds

Best known constructions:

Theorem (Idziak, Kawałek, and Krzaczkowski 2022)

Fix a modulus $m \in \mathbb{N}$ and let $r \geq 2$ be the number of its prime divisors.
AND_n can be computed by ...

- *a MOD_m-circuit of depth 2 and size $2^{\mathcal{O}(n^{1/r} \cdot \log n)}$.*



Depth-2 construction

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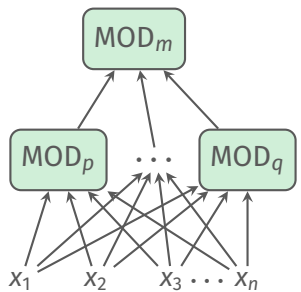
- *for every $h \in \mathbb{N}$, a MOD_m-circuit of depth $h+1$ and size $2^{\mathcal{O}(n^{1/(h\cdot(r-1))} \cdot \log n)}$.*



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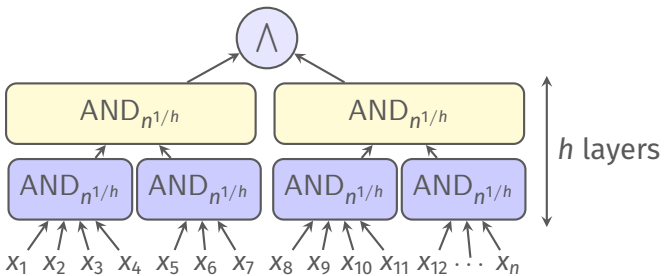
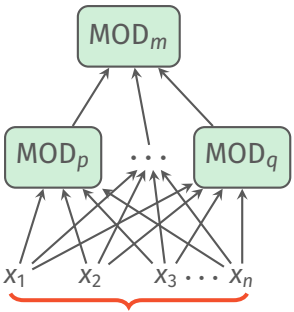


Depth-$(h+1)$ construction

Symmetries of the upper bound constructions

Both constructions are *symmetric* under different permutation groups:



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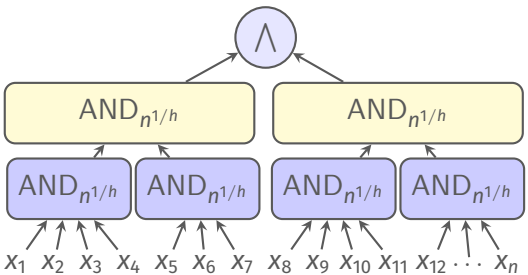

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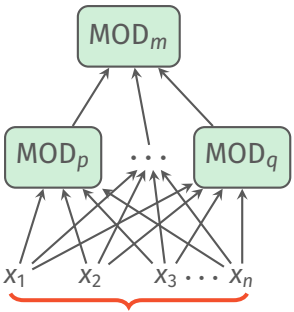

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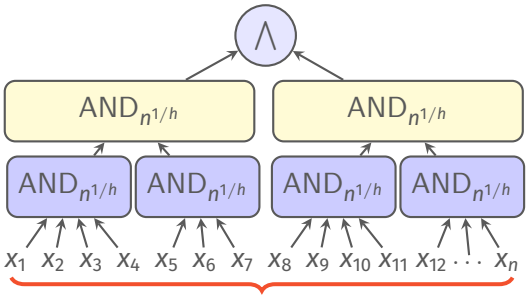

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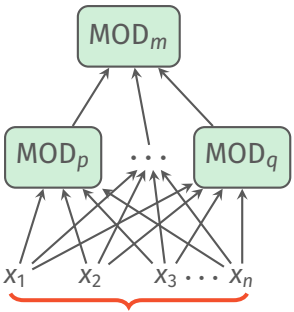

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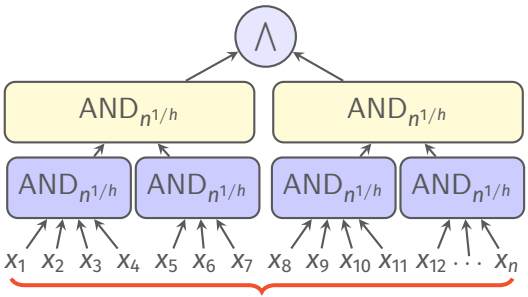

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Question: Are these circuits **size-optimal** for their respective symmetry groups?

Main result

Theorem

Fix a modulus $m \in \mathbb{N}$ and let $r \geq 2$ be the number of its prime divisors.
Let C be a MOD_m-circuit computing AND_n.

- *If C is $\mathbf{Sym}_n$-symmetric, then its size is at least $2^{\Omega(n^{1/r} \cdot \log n)}$.*
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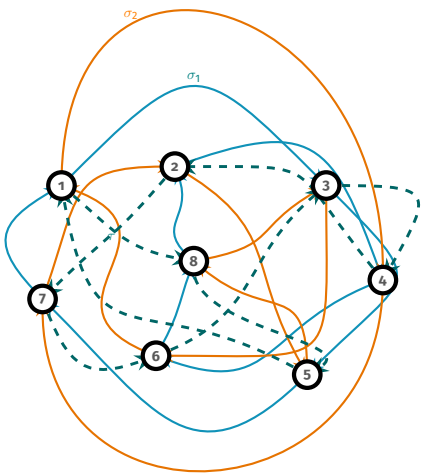
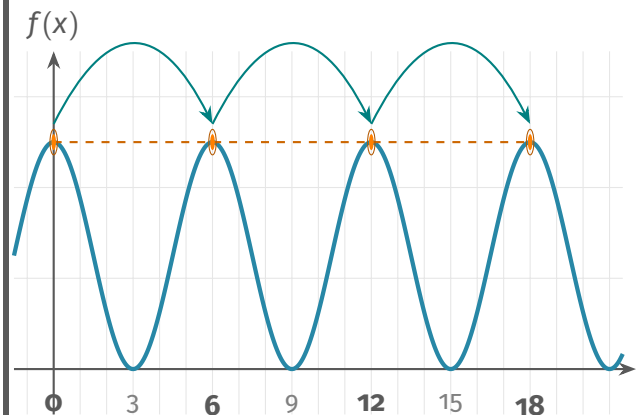
Both lower bounds are optimal, **Symmetric Exponential Size Hypothesis** confirmed.

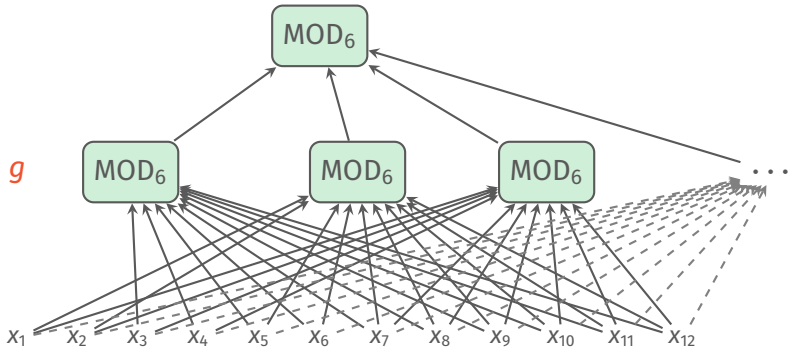
Group-theoretic support theorems



Periodicity of modular functions



$$f(0) = f(6) = f(12) = f(18) = \dots$$


MOD6
g
MOD6
MOD6
MOD6
x1
x2
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x4
x5
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x7
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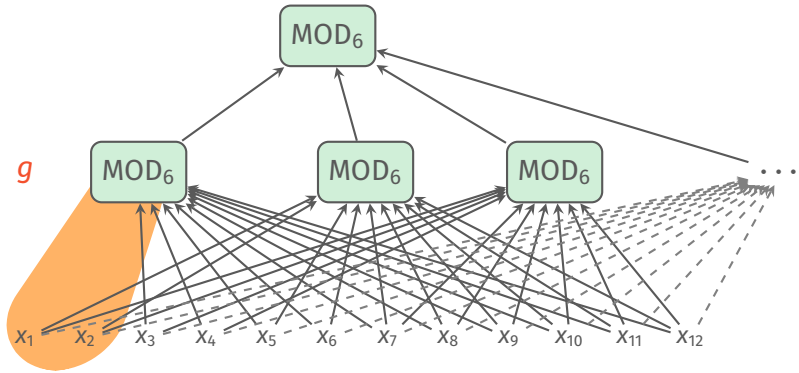
x9
x10
x11
x12



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smallest set whose stabiliser fixes g

function computed at g
depends only on # of 1s
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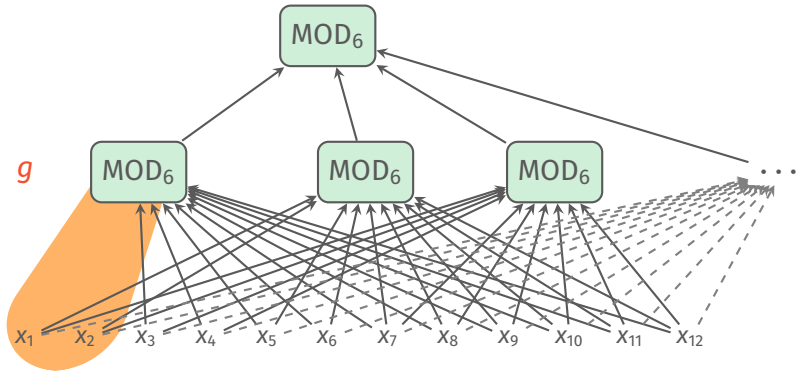

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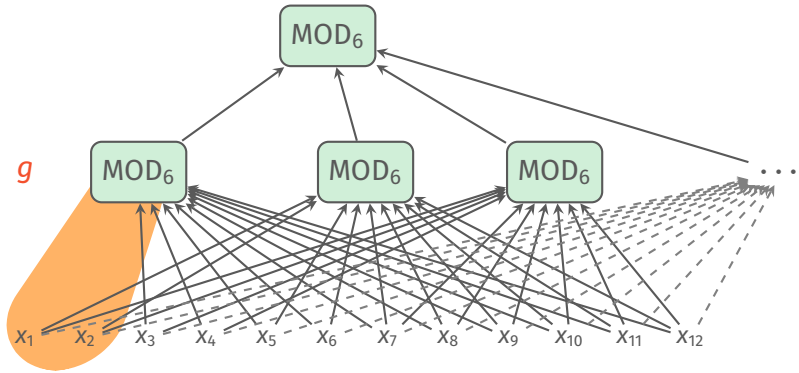

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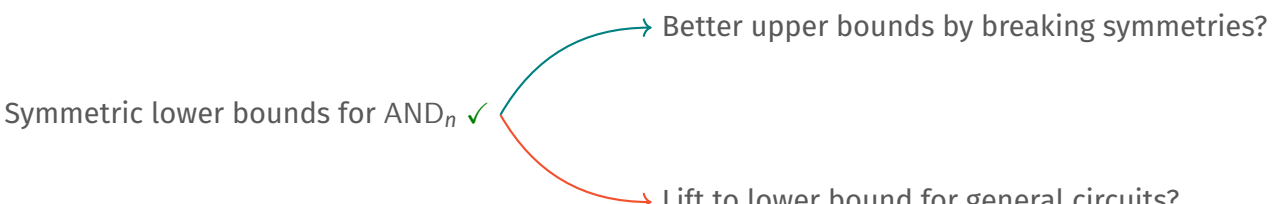
$\Rightarrow$ To compute AND_n, periods, support sizes and hence circuits must be large.

Future work

Solve CC^0 vs ACC^0!



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