# Symmetric Algebraic Circuits and Homomorphism Polynomials

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Les Houches, May 2025

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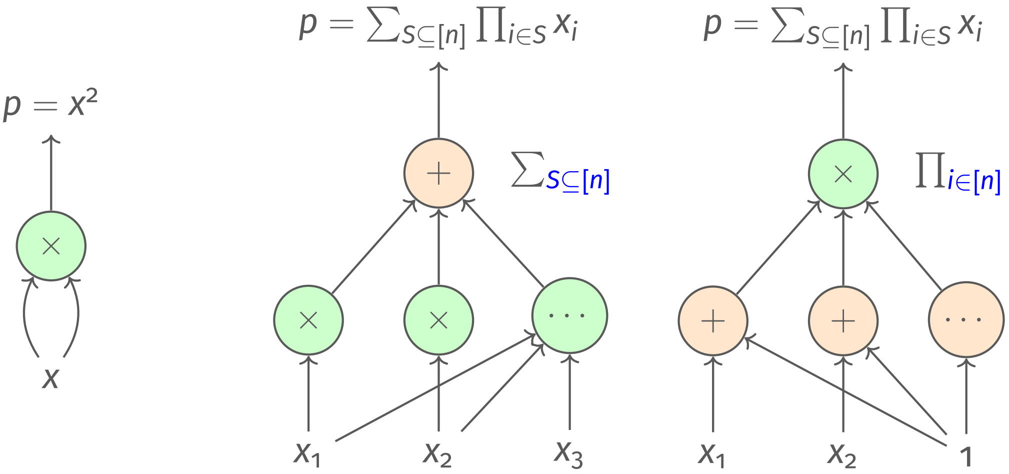
# Algebraic Circuits

The **complexity of a polynomial** $p$ is the *size* of the smallest *algebraic circuit* representing $p$.



**Determinant**

$$\det_n = \sum_{\pi \in \mathbf{Sym}_n} \operatorname{sgn}(\pi) \cdot x_{1\pi(1)} \cdots x_{n\pi(n)}$$

Circuit complexity: $\mathcal{O}(n^4)$.

**Permanent**

$$\mathrm{perm}_n = \sum_{\pi \in \mathbf{Sym}_n} x_{1\pi(1)} \cdots x_{n\pi(n)}$$

Circuit complexity: **???** (at most $\mathcal{O}(2^n \cdot n^2)$).

“VP $=$ VNP?” is the question “Does $(\mathrm{perm}_n)_{n\in\mathbb{N}}$ admit polynomial-size algebraic circuits?”

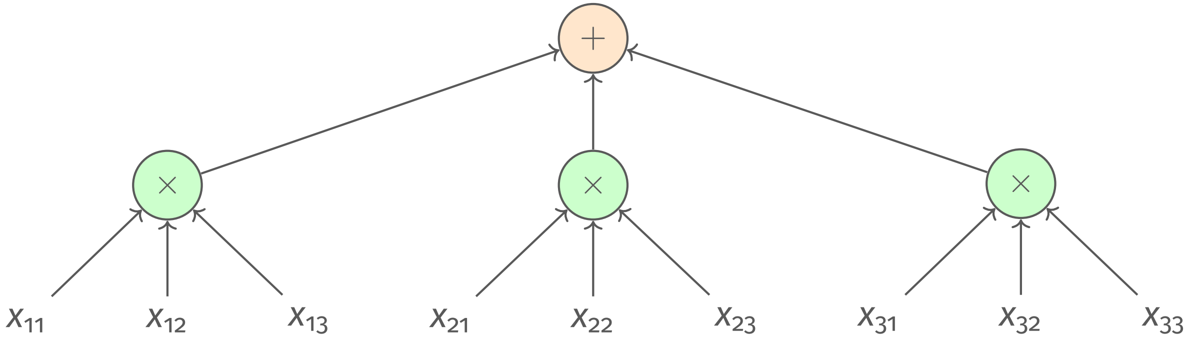
**Theorem (Dawar, Wilsenach; 2020)**

1. *There are polynomial-size symmetric circuits for* $(\mathrm{det}_n)_{n\in\mathbb{N}}$.
2. *There are no symmetric circuits for* $(\mathrm{perm}_n)_{n\in\mathbb{N}}$ *of subexponential size.*

**In this work:** Complete characterisation of polynomials with polynomial-size symmetric circuits.

# Symmetric algebraic circuits

- Let $\mathcal{X}_{n,m} := \{x_{ij} \mid i \in [n], j \in [m]\}$.
- $\mathbf{Sym}_n \times \mathbf{Sym}_m$ acts on $\mathcal{X}_{n,m}$: For $(\pi, \sigma) \in \mathbf{Sym}_n \times \mathbf{Sym}_m$, it is $(\pi, \sigma)(x_{ij}) = x_{\pi(i)\sigma(j)}$.
- An algebraic circuit $C$ over $\mathcal{X}_{n,m}$ is $\mathbf{Sym}_n \times \mathbf{Sym}_m$-symmetric if the action on $\mathcal{X}_{n,m}$ extends to automorphisms of $C$.



- Every $\mathbf{Sym}_n \times \mathbf{Sym}_m$-symmetric polynomial $p \in \mathbb{Q}[\mathcal{X}_{n,m}]$ defines a $\mathbb{Q}$-valued function on *bipartite $(n, m)$-vertex graphs*.
- **Example:** $\mathrm{perm}_n(G)$ is the number of perfect matchings in a bipartite $(n, n)$-vertex graph $G$.

**Fact**

*Every* $\mathbf{Sym}_n \times \mathbf{Sym}_m$*-symmetric polynomial can be written as a linear combination of* ***homomorphism*** *or* ***subgraph polynomials****.*

Let $F$ be a bipartite graph. For each $n, m \in \mathbb{N}$, the following polynomial evaluated in an $(n, m)$-vertex graph $G$ counts the number of homomorphisms from $F$ to $G$.

$$\mathrm{hom}_{F,n,m} := \sum_{h \colon V(F) \to [n] \uplus [m]} \prod_{ab \in E(F)} x_{h(a)h(b)}.$$

Let $\mathfrak{T}^k_{n,m}$ be the set of $\mathbb{Q}$-linear combinations of polynomials $\text{hom}_{F_i,n,m}$ where all $F_i$ have *treewidth at most $k$*.

**Theorem**

*For every family of polynomials $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$, the following are equivalent:*

1. *there exists a constant $k \in \mathbb{N}$ such that $p_{n,m} \in \mathfrak{T}^k_{n,m}$ for all $n, m \in \mathbb{N}$,*
2. *the $p_{n,m}$ admit $\mathbf{Sym}_n \times \mathbf{Sym}_m$-symmetric circuits of sizeorbit size polynomial in $n + m$.*

**Theorem (Curticapean, Dell, Marx; 2017)**

*The problem of computing linear combinations of induced subgraph counts, parametrized by their size, is*

- *in* FPT, *if it is expressible as a linear combination of homomorphism counts from bounded-treewidth graphs,*
- #W[1]-*hard, otherwise.*

**Differences to our result:**

- Syntactic complexity of polynomials vs computational complexity of the counting functions.
- Our pattern graphs have no fixed size.
- Our lower bound is unconditional.

**Theorem**

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1. *there exists a constant $k \in \mathbb{N}$ such that $p_{n,m} \in \mathfrak{T}^k_{n,m}$ for all $n, m \in \mathbb{N}$,*
2. *the $p_{n,m}$ admit $\mathbf{Sym}_n \times \mathbf{Sym}_m$-symmetric circuits of orbit size polynomial in $n + m$.*

**Problem:** When is a polynomial (not) in $\mathfrak{T}^k_{n,m}$?

- If $p_{n,m}$ is the *subgraph count polynomial* of a sublinear-size graph $F_{n,m}$, then $p_{n,m}$ is tractable iff $\text{vc}(F_{n,m})$ is bounded by a constant.
- **Conjecture:** In general, $\min\{\text{vc}(F), \text{vc}(\overline{F})\}$ is the criterion for tractability of subgraph polynomials.

**Theorem**

*If $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$ admits poly-size symmetric circuits, then the counting width (Weisfeiler-Leman dimension) of the function defined by $p_{n,m}$ on bipartite graphs is bounded.*

**Open question:** Is the converse true?

# Summary

- With respect to symmetric circuits, $\mathrm{VP} \neq \mathrm{VNP}$ (Dawar, Wilsenach; 2020).
- "Symmetric VP" can be characterised as the class of all polynomials expressible via *bounded-treewidth homomorphism counts*.
- In special cases, such as subgraph polynomials, this translates to explicit criteria for super-polynomial lower bounds.
- **Application:** A (conditional) complexity dichotomy for the *immanant* polynomials due to Curticapean (2021) holds unconditionally for symmetric circuits.